Recent progress on the modular isomorphism problem

TARO SAKURAI (櫻井 太朗)

May 29, 2026, SUSTech, Shenzhen, CHINA

Representations of Finite Groups and Ramifications

A Conference in Honor of Michel Broué's 80th Birthday and His Fundamental Contributions

CHIBA UNIVERSITY

Contents

Contents

Introduction — Notation

- p: a prime
- $\mathbb{F}$: a field of characteristic p
- G, H: finite p-groups
- $[G,G]$: the commutator subgroup
- $Z(G)$: the center
- $d(G)$: the minimum number of generators
- D_n: the dihedral group of order n

Introduction — Problem

There was a long-standing open problem on the group algebra $\mathbb{F}G$ of a finite p-group G over a field $\mathbb{F}$ of positive characteristic p.

Problem 1.1 (Modular isomorphism problem)

If $\mathbb{F}G \cong \mathbb{F}H$, then $G \cong H$?

Introduction — Positive results

It is known that $\mathbb{F}G \cong \mathbb{F}H$ implies $G \cong H$ in the following cases, if $\mathbb{F} = \mathbb{F}_p$.

1. $|G| \mid p^5$ (Salim–Sandling 1996)
2. $|G| \mid 3^7$ (Margolis–Moede 2020)
3. $|G| \mid 2^8$ (Eick 2008, Margolis–Moede 2020)

From now on, we focus on results that hold for all fields, not only $\mathbb{F} = \mathbb{F}_p$. The strongest results are only available for $\mathbb{F} = \mathbb{F}_p$.
For instance,

- $|G| \mid 2^6$ (Margolis–S. 2026?)

Introduction — Positive results

It is known that $\mathbb{F}G \cong \mathbb{F}H$ implies $G \cong H$ in the following cases.

1. Abelian groups (Deskins 1956)
2. 2-groups of maximal class (Carlson 1979)
3. $|G : Z(G)| = p^2$ (Drensky 1989)
4. Metacyclic groups (Sandling 1996, Margolis–S. 2025)
5. $d(G) = 2$ and $[G,G] \leq Z(G)$
   (Broche–del Río 2021, Margolis–S. 2025)

$\vdots$

Introduction — Negative results

However, the counterexamples are found for $p = 2$.

Theorem 1.2 (García-Lucas–Margolis–del Río*)

Let $p = 2$, $n > m > 2$ and

$$G \cong \left\langle x, y, z \;\middle|\; \begin{array}{l} x^{2^n} = 1,\ y^{2^m=1},\ z^4 = 1, \\ [y,x] = z,\ z^x = z^{-1},\ z^y = z^{-1} \end{array} \right\rangle,$$

$$H \cong \left\langle a, b, c \;\middle|\; \begin{array}{l} a^{2^n} = 1,\ b^{2^m} = 1,\ c^4 = 1, \\ [b,a] = c,\ c^a = c^{-1},\ c^b = c \end{array} \right\rangle.$$

Then $\mathbb{F}G \cong \mathbb{F}H$ but $G \not\cong H$.

In the smallest case, the groups are of order $2^{4+3+2} = 2^9 = 512$.

*D. García-Lucas, L. Margolis and Á. del Río, 'Non-isomorphic 2-groups with isomorphic modular group algebras', *J. Reine Angew. Math.* 783 (2022) 269–274.

Introduction — Defect groups

The counterexamples tell us that Morita equivalent blocks may have non-isomorphic defect groups over a field.

Corollary 1.3

Let G and H be the counterexamples (finite 2-groups).
Then $\operatorname{mod} B_0(\mathbb{F}G) \simeq \operatorname{mod} B_0(\mathbb{F}H)$ but $G \not\cong H$.

This is in sharp contrast to the (p-adic) integer case.
(ROGGENKAMP–SCOTT 1987, WEISS 1988)

Contents

New invariants — Abelian subquotients

The following are classic results.

Theorem 2.1

- $\mathbb{F}G \cong \mathbb{F}H \implies Z(G) \cong Z(H)$.
- $\mathbb{F}G \cong \mathbb{F}H \implies G/[G,G] \cong H/[H,H]$.
- $\mathbb{F}G \cong \mathbb{F}H \implies Z(G) \cap [G,G] \cong Z(H) \cap [H,H]$.

We use similar arguments and prove, among other things, the following (as a special case).

Theorem 2.2 (Margolis–S.–Stanojkovski*)

$$\mathbb{F}G \cong \mathbb{F}H \implies G/Z(G)[G,G] \cong H/Z(H)[H,H].$$

* L. Margolis, T. Sakurai and M. Stanojkovski, 'Abelian invariants and a reduction theorem for the modular isomorphism problem', *J. Algebra* 636 (2023) 1–27.

New invariants — Positive results

As an application of the previous theorem, we prove the following.

Theorem 2.3 (Margolis–S.–Stanojkovski*)

Assume that $d(Z(G)) = 1$ and $G/Z(G)$ is dihedral. Then

$$\mathbb{F}G \cong \mathbb{F}H \implies G \cong H.$$

Remark 2.4

The central quotients of the counterexamples are D_8.

* L. Margolis, T. Sakurai and M. Stanojkovski, 'Abelian invariants and a reduction theorem for the modular isomorphism problem', *J. Algebra* 636 (2023) 1–27.

New invariants — Classification

We use the following classification to prove the previous theorem.

Theorem 2.5 (MARGOLIS–S.–STANOJKOVSKI*)

Assume that $d(Z(G)) = 1$ and $G/Z(G)$ is dihedral. Then G is isomorphic to one of the following for some $m \geq 1$ and $n \geq 2$.

$$\mathrm{D}_{2^m|n} = \langle a, b, c \mid a^2 = 1,\ b^2 = 1,\ (ab)^{2^{n-1}} = c^{2^{m-1}},\ \ (\textit{common})\rangle,$$

$$\mathrm{Q}_{2^m|n} = \langle a, b, c \mid a^2 = c,\ b^2 = c,\ (ab)^{2^{n-1}} = c^{2^{m-1}+2^{n-1}},\ (\textit{common})\rangle,$$

$$\mathrm{S}_{2^m|n} = \langle a, b, c \mid a^2 = 1,\ b^2 = c,\ (ab)^{2^{n-1}} = c^{2^{m-1}+2^{n-2}},\ (\textit{common})\rangle,$$

where $(\textit{common}) = c^{2^m} = [a, c] = [b, c] = 1$.

These groups are finite 2-groups of maximal class if $m = 1$.

* L. MARGOLIS, T. SAKURAI and M. STANOJKOVSKI, 'Abelian invariants and a reduction theorem for the modular isomorphism problem', *J. Algebra* 636 (2023) 1–27.

Contents

Tensor factors — Elementary abelian

Theorem 3.1 (Margolis–S.–Stanojkovski*)

Assume that E is a finite elementary abelian p-group.

$$\mathbb{F}[G \times E] \cong \mathbb{F}[H \times E] \implies \mathbb{F}G \cong \mathbb{F}H.$$

Recall that $\mathbb{F}[G \times E] \cong \mathbb{F}G \otimes_{\mathbb{F}} \mathbb{F}E$. This shows that it is possible to kill certain tensor factors.

---

* L. Margolis, T. Sakurai and M. Stanojkovski, 'Abelian invariants and a reduction theorem for the modular isomorphism problem', *J. Algebra* 636 (2023) 1–27.

Tensor factors — Indecomposable

This raises the following question.

Do tensor factorizations of algebras always come from direct decompositions of groups?

We do not know even the simplest case.

Question 3.2 (Carlson–Kovács*)

If G is directly indecomposable, then $\mathbb{F}G$ is tensor indecomposable?

Theorem 3.3 (García-Lucas–del Río–S.†)

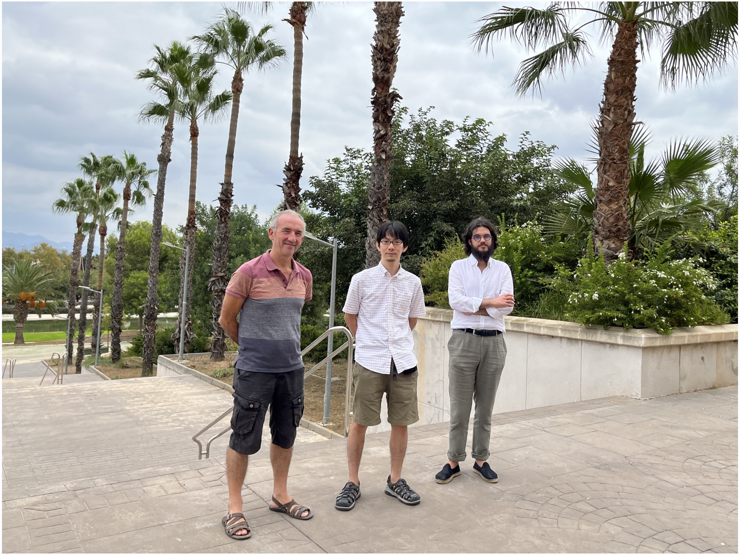
Assume that $d(G) \leq 3$ and $\mathbb{F} = \mathbb{F}_p$.
If G is directly indecomposable, then $\mathbb{F}G$ is tensor indecomposable.

---

* J.F. Carlson and L.G. Kovács, 'Tensor factorizations of group algebras and modules', *J. Algebra* 175 (1995) 385–407.

† D. García-Lucas, Á. del Río and T. Sakurai, 'On commutative tensor factors of group algebras', *Algebr. Represent. Theory* 29 (2026) 153–160.

Ángel del Río & Diego García-Lucas — Murcia



Contents

Counterexamples — Identification

Theorem 4.1 (MARGOLIS–S.*)

Let $p = 2$ and assume that $d(G) = 2$ and $G/Z(G)$ is non-abelian dihedral. Then $\mathbb{F}G \cong \mathbb{F}H$ but $G \not\cong H$ if and only if

$$G \cong \left\langle x, y, z \;\middle|\; \begin{array}{l} x^{2^n} = 1,\ y^{2^m} = 1,\ z^{2^\ell} = 1, \\ [y,x] = z,\ [z,x] = z^{-2},\ [z,y] = z^{-2} \end{array} \right\rangle,$$

$$H \cong \left\langle a, b, c \;\middle|\; \begin{array}{l} a^{2^n} = 1,\ b^{2^m} = a^{2^m},\ c^{2^\ell} = 1, \\ [b,a] = c,\ [c,a] = c^{-2},\ [c,b] = c^{-2} \end{array} \right\rangle,$$

up to exchange, for some $n > m > \ell \geq 2$.

---

[*] L. MARGOLIS and T. SAKURAI, 'Identification of non-isomorphic 2-groups with dihedral central quotient and isomorphic modular group algebras', *Rev. Mat. Iberoam.* 41 (2025) 1973–2002.

Counterexamples — Remarks

The first counterexamples correspond to the case $\ell = 2$. In particular, we also find new counterexamples. Furthermore, algebra isomorphisms and proofs become simpler than the original ones.

These groups themselves are also found by BAGIŃSKI–ZABIELSKI*, independently. The algebra isomorphisms are, however, established only for $\mathbb{F} = \mathbb{F}_2$ (formally).

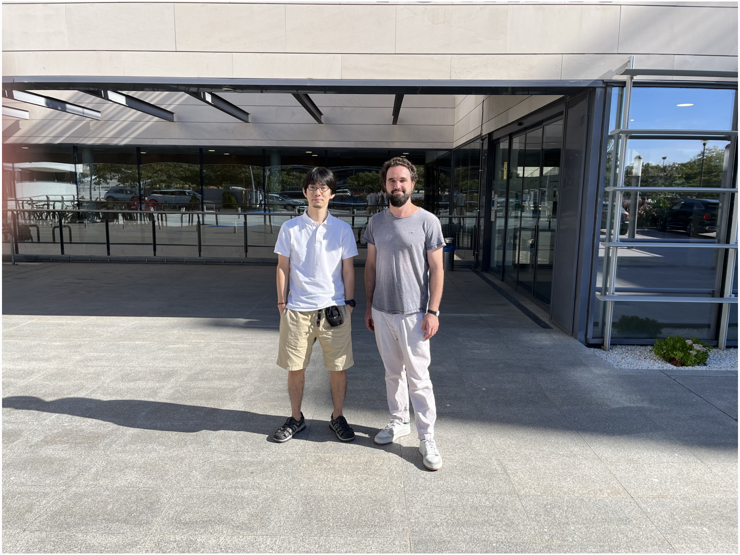
*C. BAGIŃSKI and K. ZABIELSKI, 'The modular isomorphism problem: the alternative perspective on counterexamples', *J. Pure Appl. Algebra* 229 (2025) 107826.

Counterexamples — Classification

We classified finite 2-groups G that $d(G) = 2$ and $G/Z(G)$ is non-abelian dihedral to prove the previous theorem.

$$
\begin{tikzcd}[column sep=small]
2^{n+m+2} & G_{\text{⚂}}(n,m,2) & G_{\text{⚀}}(n,m,2) & G_{\text{⚄}}(n,m,2) & G_{\text{⚃}}(n,m,2) & G_{\text{⚁}}(n,m,2) & G_{\text{⚅}}(n,m,2) \\
2^{n+m+3} & G_{\text{⚂}}(n,m,3) \arrow[ur, two heads] & G_{\text{⚀}}(n,m,3) \arrow[u, two heads] & G_{\text{⚄}}(n,m,3) \arrow[ul, two heads] & G_{\text{⚃}}(n,m,3) \arrow[ur, two heads] & G_{\text{⚁}}(n,m,3) \arrow[u, two heads] & G_{\text{⚅}}(n,m,3) \arrow[ul, two heads] \\
\vdots & \vdots \arrow[ur, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[ul, two heads] & \vdots \arrow[ur, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[ul, two heads] \\
2^{n+m+\ell-1} & G_{\text{⚂}}(n,m,\ell-1) \arrow[ur, two heads] & G_{\text{⚀}}(n,m,\ell-1) \arrow[u, two heads] & G_{\text{⚄}}(n,m,\ell-1) \arrow[ul, two heads] & G_{\text{⚃}}(n,m,\ell-1) \arrow[ur, two heads] & G_{\text{⚁}}(n,m,\ell-1) \arrow[u, two heads] & G_{\text{⚅}}(n,m,\ell-1) \arrow[ul, two heads] \\
2^{n+m+\ell} & G_{\text{⚂}}(n,m,\ell) \arrow[ur, two heads] & G_{\text{⚀}}(n,m,\ell) \arrow[u, two heads] & G_{\text{⚄}}(n,m,\ell) \arrow[ul, two heads] & G_{\text{⚃}}(n,m,\ell) \arrow[ur, two heads] & G_{\text{⚁}}(n,m,\ell) \arrow[u, two heads] & G_{\text{⚅}}(n,m,\ell) \arrow[ul, two heads] \\
\vdots & \vdots \arrow[ur, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[ul, two heads] & \vdots \arrow[ur, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[ul, two heads]
\end{tikzcd}
$$

Leo Margolis — Madrid



Counterexamples — Open problem

The modular isomorphism problem remains open for odd primes.

Open problem 4.2

Assume that $p > 2$. If $\mathbb{F}G \cong \mathbb{F}H$, then $G \cong H$?

Summary

1. Modular isomorphism problem: $\mathbb{F}G \cong \mathbb{F}H \implies G \cong H$?
2. New invariants: $\mathbb{F}G \cong \mathbb{F}H \implies G/Z(G)[G,G] \cong H/Z(H)[H,H]$.
3. Positive results when $d(Z(G)) = 1$ and $G/Z(G)$ is dihedral.
4. Reduction theorem for elementary abelian direct factors.
5. Tensor indecomposability of $\mathbb{F}G$ when $d(G) \leq 3$ and $\mathbb{F} = \mathbb{F}_p$.
6. Studied groups with $d(G) = 2$ and $G/Z(G)$ is dihedral.
7. Complete answer to the problem within the above class.
8. New counterexamples to the modular isomorphism problem.